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A Fast and Accurate Numerical Approach for Pricing American-Style Power Options

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Abstract: In this paper, we present a fast and accurate numerical approach applied to specific American-style derivatives, namely American power call and put options, whose main feature is that the underlying asset is raised to a power. The study is set in the Black–Scholes framework, and we consider continuously paying dividends assets and arbitrary positive values for the power. It is important to note that although a log-normal process raised to a power is again log-normal, the resulting change in variables may lead to a negative dividend rate, and this case remains largely understudied in the literature. We derive closed-form formulas for the perpetual options' optimal boundaries and for the fair prices. For finite maturities, we approximate the optimal boundary using some first-hitting properties of the Brownian motion. As a consequence, we obtain the option price quickly and with relatively high accuracy—the error is at the third decimal position. We further provide a comprehensive analysis of the impact of the parameters on the options' value, and discuss ordinary European and American capped options. Various numerical examples are provided.

Keywords: American power options; optimal boundaries; perpetual options; finite maturity options; capped options

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1. Introduction

American-style derivatives, especially American-style options, are some of the most traded instruments in modern financial markets. Their main feature is the holder's right to early exercise, which makes them useful both for hedging purposes and for additional profits. However, the existing variety of financial risks has motivated investors to seek new opportunities to protect themselves from market uncertainty. Thus, in recent years, in addition to the large number of non-ordinary (exotic) options, many new instruments and financial strategies have emerged, inspiring researchers to explore their properties and invent methods for their evaluation. The following is a non-exhaustive list covering some of the most recent developments in the field: American options under stochastic volatility [1]; numerical evaluation of options under Heston's model [2]; generalized American derivatives leading to one- or two-sided optimal stopping problems [3–5]; financial instruments leading to first-exit tasks [7,8]; American put options written on zero-coupon bonds [9]; novel numerical methods for pricing American options [10–13]; a projection

and contraction approach for pricing different kinds of American options [14,15]; critical analysis of Monte Carlo simulations for pricing American options [16]; vulnerable American options without maturity constraints [17]; financial derivatives leading to double continuation regions [18,19]; the hedging task for American options in the presence of transaction costs [20]; acceleration clauses reducing the option's life [21]; American barrier options in light of the first time digitals [22]; a Monte Carlo simulation approach for pricing American time-capped options [23].

We add to this growing literature by investigating and proving new properties of American power options. Unlike ordinary options, the underlying asset of a power option is raised to a power, so that investors may hedge different non-linear financial risks thanks to the non-linear payoff function, see, e.g., [24]. Their utility can be understood as follows: an ordinary option pays its holder the overprice (call) or the underprice (put); if the investor wants relatively larger compensation when the underlying asset falls deeply in-the-money, then powers greater than one can provide such payoffs—the higher the power, the greater the increase in payment. On the other hand, the option's writer may prefer to pay proportionally less when the underlying asset is deeply in-the-money, and powers less than one lead to such payoffs—the lower the power, the smaller the increase in payment.

Power options are actually traded financial instruments and have been studied in some detail under different assumptions. For example, European-style power options in the Black–Scholes setting are investigated by [25], while stochastic volatility models are examined by [26–28], and some power options written on assets driven by Lévy jump processes and under Heston's assumptions are explored in [28]. The authors in [29] consider European-style power options under uncertain differential equations, whereas [30,31] apply the Mellin transform method for pricing American power put options. More recently, [32] has presented a fast and efficient finite difference method under the non-dividend assumptions, and [33,34] have examined the power version of both European and American exchange options.

Our study of American power options is based on the hypothesis that the underlying asset is described by a log-normal process, assuming continuously paid dividends. Although the power of a log-normal process is again log-normal, some difficulties arise. In particular, if the original model is driven by the risk-free rate r , volatility σ , and dividend δ , then raising it to the n th power leads to a model with parameters $\bar{r} = r$, $\bar{\sigma} = n\sigma$, and $\bar{\delta} = n\delta - (n-1)(r + n\sigma^2/2)$. Note that the initial and the new risk-free rates have to be equal due to some risk-neutral considerations, which implies that for some parameter values the dividend rate $\bar{\delta}$ is negative. Therefore, it is not possible to study power options through a simple change in parameters, which explains why these models remain largely understudied (we refer to [18,35] for some examples).

Our approach for dealing with dividends is to introduce an additional discount factor, following [36]. Using this transformation, we obtain the shape of the optimal regions and the corresponding early exercise boundaries. In particular, we prove that if a point is optimal for a call/put option, then all points above/below it are also optimal. This suggests that the state space can be divided into two parts by a single boundary, namely the early exercise or optimal boundary, so that for a call/put option, immediate exercise will be optimal for all points above/below it. Moreover, we show that the boundary is increasing/decreasing with respect to the time to maturity for call/put options. A well-known fact for ordinary call options is that early exercise is never optimal when there are no dividends, i.e., the additional discounting factor is zero. In that case, the optimal boundary is infinitely large, which we show is also true for power call options, but only for powers greater than one and for small enough discount factors (dividends). We prove that this limit is zero when the power is equal to one, and finite when the power is strictly

less than one. In addition, we study perpetual options for which there are no maturity constraints and show that the optimal boundary is time-independent, deriving closed-form expressions for it and for the fair option price.

We further consider the problem of approximating the optimal boundary when the time horizon is finite. We first obtain its value when the time to maturity tends to zero, and then derive the value at each subsequent node in a time partition that maximizes the holder's financial utility. It turns out that three or four points are quite enough for a precise approximation, since the estimated option price is very close to the true one, with error appearing in the third decimal place. For greater accuracy, we may approximate the boundary on a denser grid, in which case the free boundary differential equation that describes the American option pricing task can be viewed as a boundary value problem stated in a known region, to which we apply a modification of the Crank–Nicolson finite difference approach. We provide various numerical experiments to illustrate and validate the obtained theoretical results.

Finally, we discuss the so-called capped call/put options, where exercise cannot take place at a price higher/below than some predetermined level. It turns out that results for capped non-power options carry over to power options, e.g., the optimal boundary for a capped power call/put option is the minimum/maximum of the boundary of the corresponding uncapped option and the cap level.

The paper is organized as follows. In Section 2, we define our model and derive the shape of the exercise regions. Perpetual options are examined in Section 3, while the corresponding finite maturity options are considered in Section 4. We present several numerical examples in Section 5. Some evidence for the S&P 500 Index during the 2008 global financial crisis is discussed in Section 6. Supplementary results are collected in Appendices A–C.

2. Exercise Regions

Suppose that the underlying asset is modeled by the log-normal process

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right), \quad (1)$$

under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral measure and B_t defines a Brownian motion. The constant r is the risk-free rate, and we denote by $\lambda \geq 0$ the additional discount factor. We assume that $r + \lambda > 0$, but negative values for r are still possible. We denote by $\mathcal{A}$ the infinitesimal generator of the process (1) and by $\mathcal{B}$ the associated differential operator:

$$\begin{aligned} (\mathcal{A}f)(x) &= rx f'(x) + \frac{\sigma^2 x^2}{2} f''(x), \\ (\mathcal{B}g)(x) &= (\mathcal{A}g)(x) - (r + \lambda)g(x). \end{aligned} \quad (2)$$

Let $T \leq \infty$ be the maturity date, and denote by $n > 0$ the power to which the underlying asset is raised. The payoff functions are given by

$$\begin{aligned} N(t, x) &= e^{-\lambda t} G(x) = e^{-\lambda t} (x^n - K^n)^+ && \text{for call,} \\ N(t, x) &= e^{-\lambda t} G(x) = e^{-\lambda t} (K^n - x^n)^+ && \text{for put,} \end{aligned} \quad (3)$$

where we view the constant K as a strike. Sometimes, we shall parametrize N with respect to the time to maturity, $\tau := T - t$, and we shall write $(x; \tau)$ instead of (t, x) . This is possible due to the Markovian nature of process (1), which allows us to assume that the initial time

is zero. We will denote the option price by $P(t, x)$ or $P(x; \tau)$, respectively. The proposition below states that we can think of the additional discount factor as a dividend rate.

Proposition 1. *By an (r, λ, δ) -model, we shall mean a financial market model that consists of an underlying asset paying continuous dividends at rate δ , a risk-free asset with rate of return r , and an option discounted by the factor λ . Under these assumptions, an (r, λ, δ) -model is equivalent to an $(r - \delta, \lambda + \delta, 0)$ -one.*

Proof. See Proposition 2.3 in [37]. $\square$

Remark 1. Proposition 1 shows that the conditions $\lambda \geq 0$ and $r + \lambda > 0$ lead to both a positive risk-free rate and a non-negative dividend rate for a regular option model (i.e., without discounting).

The price function of an American power option can be written as

$$P(t, x) = \sup_{\zeta \in \mathcal{T}_{t,T}} \mathbb{E}^{t,x} \left[e^{-r(\zeta-t)} N(\zeta, S_\zeta) \right],$$

where the supremum is taken over $\mathcal{T}_{t,T}$, the set of all stopping times with values between t and T . Thus, we can define the associated optimal points as follows:

Definition 1. A point (t, x) is optimal for the holder if $P(t, x) = N(t, x)$.

Definition 1 formalizes mathematically the idea that it is optimal for the holder to exercise the option at time t at a spot price x if the amount he will receive, $N(t, x)$, is greater than the financial outcome of any other strategy (stopping time), $\mathbb{E}^{t,x} [e^{-r(\zeta-t)} N(\zeta, S_\zeta)]$. On the other hand, we will say that the point $(x; \tau)$ is optimal with respect to the time-to-maturity parametrization when the point $(T - \tau, x)$ is optimal with respect to the original one. A necessary condition for optimality is presented in the following proposition.

Proposition 2. *If a point (t, x) is optimal, then $(\mathcal{B}G)(x) < 0$.*

Proof. Note that the inequality $(\mathcal{B}G)(x) < 0$ is equivalent to the variational inequality

$$N_t(t, x) + \mathcal{A}N(t, x) - rN(t, x) < 0,$$

which holds in the optimal region. $\square$

The shape of the optimal regions is obtained in the following proposition.

Proposition 3. *Suppose that $(x; \tau)$ is an optimal point. If we have a call option and $y > x$, then the point $(y; \tau)$ is also optimal. Otherwise, if we have a put option and $y < x$, then the point $(y; \tau)$ is also optimal. Moreover, the optimal boundary of a call option increases with respect to the time to maturity, while the put one decreases.*

Proof. We will consider only the put case, since the call one is symmetric. First, note that if a point $(x; \tau_1)$ (parametrized with respect to the time to maturity) is optimal, then $(x; \tau_2)$ is also optimal, for every $\tau_2 < \tau_1$. This is true because longer maturities provide richer sets of financial strategies for the option's holder.

Suppose now that, for some $y < x$, the point $(x; \tau)$ is optimal, but $(y; \tau)$ is not. Let ζ^x be the first hitting moment of x by the asset, starting from y . For an arbitrary stopping time $\bar{\zeta}$, we shall denote by ζ the strategy $\zeta = \bar{\zeta} \wedge \zeta^x \wedge \tau$. This strategy gives no worse financial outcome than $\bar{\zeta}$ because $(x; \bar{\tau})$ is an optimal point for all $\bar{\tau} < \tau$. Also, $(\mathcal{B}G)(S_u(\omega)) < 0$ for

every u such that $u < \zeta(\omega)$, because for such sample paths ω , the inequality $S_u(\omega) < x$ holds, see Lemma A1. It now follows from Dynkin's formula that

$$\mathbb{E}^y \left[e^{-(r+\lambda)\bar{\zeta}} G(S_{\bar{\zeta}}) \right] - G(y) \leq \mathbb{E}^y \left[e^{-(r+\lambda)\zeta} G(S_{\zeta}) \right] - G(y) = \mathbb{E}^y \left[\int_0^{\zeta} (\mathcal{B}G)(S_u) du \right] < 0.$$

Hence, immediate exercise is the holder's best strategy. This completes the proof. $\square$

Given Proposition 3, we conclude that the state space can be divided into two connected parts—all points in the first set are optimal, while in the second set keeping the option is preferable. Moreover, these sets are separated by a time-function, so that for call/put options the optimal one is above/below it. This function is also known as the optimal boundary and we will denote it by $c(\tau)$, where τ is the time to maturity.

Next, we obtain the values of the optimal boundary near the expiration as well as in the perpetual case, i.e., $c(0)$ and $c(\infty)$. Later, we will use these values to approximate the whole boundary.

2.1. The Boundary Value at Maturity

First, we discuss the boundary value when $\tau = 0$. An important role is played by the constant L , defined as

$$L := (n-1) \left(r + \frac{\sigma^2}{2} n \right) - \lambda. \quad (4)$$

The following two propositions determine the initial boundary value.

Proposition 4 (Call). *Consider a call option. If $L \geq 0$, then $c(0) = \infty$. Otherwise, if $L < 0$, then*

$$c(0) = \bar{K} := K \left(\max \left(1, -\frac{r+\lambda}{L} \right) \right)^{\frac{1}{n}}.$$

Proof. Applying the differential operator $\mathcal{B}$ defined in (2) to the call payoff $G(x)$ above the strike, gives

$$(\mathcal{B}G)(x) = Lx^n + K^n(r+\lambda).$$

If $L \geq 0$, then $(\mathcal{B}G)(x) > 0$, so it follows from Proposition 2 that $c(0) = \infty$. Suppose now that $L < 0$. Obviously, $c(0) \geq K$ because the call option pays nothing below the strike. On the other hand, $c(0) \geq K \left(-\frac{r+\lambda}{L} \right)^{1/n}$ by Proposition 2. Suppose that there is a point $x > \bar{K}$ belonging to the continuation region near maturity. Using that $P(t, x) > N(t, x)$ for all x in this set and the fact that $P(t, x)$ solves the Black–Scholes equation, $P_t + \mathcal{A}P = rP$, we get

$$\begin{aligned} 0 &< \lim_{t \rightarrow T} \frac{P(t, x) - N(t, x)}{T - t} \\ &= -\lim_{t \rightarrow T} \frac{P(T, x) - P(t, x)}{T - t} + \lim_{t \rightarrow T} \frac{N(T, x) - N(t, x)}{T - t} \\ &= \mathcal{A}P(T, x) - rP(T, x) + N_t(T, x) \\ &= e^{-rT} [(\mathcal{B}G)(x) + (r+\lambda)G(x) - rG(x) - \lambda G(x)] < 0. \end{aligned}$$

The contradiction completes the proof. $\square$

Proposition 5 (Put). *Consider a put option. If $L \geq 0$, then $c(0) = K$. Otherwise, if $L < 0$, then*

$$c(0) = K \left(\min \left(1, -\frac{r+\lambda}{L} \right) \right)^{\frac{1}{n}}. \quad (5)$$

Proof. In this case, the operator $(\mathcal{B}G)(x)$ becomes

$$(\mathcal{B}G)(x) = -(Lx^n + K^n(r + \lambda)).$$

The largest x for which $(\mathcal{B}G)(x) < 0$ is given namely by (5). The rest of the proof is identical to that of Proposition 4. $\square$

Corollary 1. *The following statements hold.*

1. For a call option, if $L \geq 0$, then early exercise is never optimal, which is possible only when $n \geq 1$. If $L < 0$ and $r + (n - 1)\frac{\sigma^2}{2} > 0$, then $c(0) > K$.
2. For a put option, if $L < 0$ and $r + (n - 1)\frac{\sigma^2}{2} < 0$, then $c(0) < K$.

Proof. The corollary is a consequence of Lemma A2 and Propositions 3, 4 and 5. $\square$

Recall that $n = 1$ corresponds to the case of an ordinary American option. Then $L \geq 0$ is equivalent to $\lambda \leq 0$ and leads to the well-known fact that early exercise of a call is never optimal in the absence of discounting (dividends). On the other hand, if $L < 0$ (equivalently, $\lambda > 0$), then $r + (n - 1)\sigma^2/2 > 0$ if and only if $r > 0$, so we have $c(0) > K$ whenever $r > 0$. Similarly, for put options, we have $c(0) < K$ whenever $r < 0$. See Propositions 3.4 and 3.5 in [38] for more details.

2.2. Impact of the Power Value

Next, we investigate the behavior of the initial point of the optimal boundary as a function of the power n . To that end, let us define the constants γ_c , γ_p , d_1 , and d_2 by

$$\begin{aligned}\gamma_c &:= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right), \\ \gamma_p &:= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right), \\ d_1 &:= \frac{\sigma}{2} - \frac{r}{\sigma}, \\ d_2 &:= \frac{1}{\sigma} \ln\left(\frac{c}{x}\right).\end{aligned}\tag{6}$$

We have the following dependence on the power value.

Proposition 6. *Define $\bar{n} := 1 - 2r/\sigma^2$.*

1. If $r \geq \sigma^2/2$, then
 - (a) Call options: $K < c(0; n) < \infty$ for $n < \gamma_c$, and $c(0; n) = \infty$ for $n \geq \gamma_c$.
 - (b) Put options: $c(0; n) = K$ for every n .
2. If $r < \sigma^2/2$, then
 - (a) Call options: $c(0; n) = K$ for $0 < n \leq \bar{n}$, $K < c(0; n) < \infty$ for $\bar{n} < n < \gamma_c$, and $c(0; n) = \infty$ for $n \geq \gamma_c$.
 - (b) Put options: $c(0; n) < K$ for $n < \bar{n}$, and $c(0; n) = K$ for $n \geq \bar{n}$. The limit point is

$$\lim_{n \rightarrow 0} c(0; n) = Ke^{\frac{r-\frac{\sigma^2}{2}}{r+\lambda}}.\tag{7}$$

Proof. Consider L , defined in (4), as a function of the power n , i.e., $L(n)$. Then L is a quadratic function of n with a positive first coefficient, $L(0) = -(r + \lambda) < 0$, and $L(\gamma_c) = 0$ by Lemma A3. Hence, $L(n) < 0$ for $n < \gamma_c$, and $L(n) > 0$ for $n > \gamma_c$. Let us define

$$\alpha(n) := r + (n-1)\frac{\sigma^2}{2},$$

whose root is just $\bar{n}$. Then $\alpha(\gamma_c) > 0$, since

$$\alpha(\gamma_c) = \frac{1}{2} \left(r - \frac{\sigma^2}{2} + \sqrt{\left(r - \frac{\sigma^2}{2} \right)^2 + 2(r + \lambda)\sigma^2} \right).$$

Therefore, $\bar{n} < \gamma_c$. Note that the condition $r < \sigma^2/2$ is equivalent to $\bar{n} > 0$. Combining Propositions 4 and 5 with the second statement of Lemma A2, we establish the behavior of the boundaries. Finally, (7) follows from

$$\left(-\frac{r + \lambda}{L} \right)^{\frac{1}{n}} = \left(1 + \frac{n \left(r + (n-1)\frac{\sigma^2}{2} \right)}{r + \lambda - n \left(r + (n-1)\frac{\sigma^2}{2} \right)} \right)^{\frac{1}{n}}.$$

□

Remark 2. Recall that the discount factor λ can be viewed as a dividend rate. For power call options, if $n < 1$, the first statement of Lemma A2 shows that $L < 0$, so early exercise can be optimal. Otherwise, if $n \geq 1$, then this is true only for sufficiently large values of λ , namely for $\lambda > \bar{\lambda} = (n-1)(r + n\sigma^2/2)$. If we have an ordinary option, i.e., $n = 1$, then we arrive at the well-known fact that premature exercise can be optimal only for $\lambda > 0$.

2.3. Capped Options

It turns out that the results for ordinary American capped options can be extended to power options. Letting F be the cap level, the payment functions (3) for capped options become

$$\begin{aligned} G(x) &= (x^n \wedge F^n - K^n)^+ && \text{for call,} \\ G(x) &= (K^n - x^n \vee F^n)^+ && \text{for put,} \end{aligned}$$

where $\vee$ and $\wedge$ denote the maximum and minimum, respectively. The main result is contained in the following proposition.

Proposition 7. Denote the optimal boundary of the capped option by $c_{\text{capped}}(t)$. For put options, $c_{\text{capped}}(t) = c(t) \vee F$; otherwise, for call options, $c_{\text{capped}}(t) = c(t) \wedge F$.

Proof. We will look only at the put case, since the call one is symmetric. The proof of Proposition 3.1 in [39], which considers non-power options, i.e., $n = 1$, is divided into four parts, depending on whether the point (t, x) satisfies (A) $x \in (F, c(t))$, (B) $x < F$, (C) $\{x > F \vee c(t), F \leq c(0)\}$, or (D) $\{x > F \vee c(t), F > c(0)\}$. We can prove the present statement in the same way in the first three cases. However, the proof of the fourth case, which is identical only when $c(0) < K$ or, equivalently, when $L < 0$ and $r + (n-1)\sigma^2/2$, differs precisely in Equation (3.8) from [39], which for power options should be

$$\begin{aligned} \mathbb{E}^{t,x} \left[e^{-r(\zeta-t)} N(\zeta, S_\zeta) \right] &= \mathbb{E}^{t,x} \left[e^{-r(\zeta-t)} e^{-\lambda\zeta} \left(K^n - S_\zeta^n \right)^+ \right] \\ &= \mathbb{E}^{t,x} \left[e^{-r(\zeta-t)} e^{-\lambda\zeta} \left(K^n - S_\zeta^n \right) \right] + \mathbb{E}^{t,x} \left[e^{-r(\zeta-t)} e^{-\lambda\zeta} \left(S_\zeta^n - K^n \right) I_{S_\zeta \geq K} \right] \\ &\quad \pm e^{-\lambda t} (K^n - x^n) \\ &= e^{-\lambda t} (K^n - x^n) + \mathbb{E}^{t,x} \left[e^{-r(\zeta-t)} e^{-\lambda\zeta} \left(S_\zeta^n - K^n \right) I_{S_\zeta \geq K} \right] \end{aligned}$$

$$\begin{aligned}
& + \mathbb{E}^{t,x} \left[K^n \left(e^{-r(\zeta-t)-\lambda\zeta} - e^{-\lambda t} \right) \right] - \mathbb{E}^{t,x} \left[S_\zeta^n e^{-r(\zeta-t)-\lambda\zeta} \right] + x^n e^{-\lambda t} \\
& = e^{-\lambda t} (K^n - x^n) + \mathbb{E}^{t,x} \left[e^{-r(\zeta-t)} e^{-\lambda\zeta} (S_\zeta^n - K^n) I_{S_\zeta \geq K} \right] \\
& + \mathbb{E}^{t,x} \left[K^n \int_t^\zeta -(r+\lambda) e^{-\lambda u-r(u-t)} du \right] - x^n e^{-\lambda t} \\
& - \mathbb{E}^{t,x} \left[\int_t^\zeta L e^{-r(u-t)-\lambda u} S_u^n du \right] + x^n e^{-\lambda t} \\
& = e^{-\lambda t} (K^n - x^n) + \mathbb{E}^{t,x} \left[e^{-r(\zeta-t)} e^{-\lambda\zeta} (S_\zeta^n - K^n) I_{S_\zeta \geq K} \right] \\
& + \mathbb{E}^{t,x} \left[\int_t^\zeta e^{-r(u-t)-\lambda u} (-LS_u^n - (r+\lambda)K^n) du \right] \\
& > e^{-\lambda t} (K^n - x^n),
\end{aligned}$$

where ζ denotes the minimum between the first hitting time of F and the maturity date T . Note that we have used Proposition 5 and Dynkin's formula above, keeping in mind that $-LS_u^n - (r+\lambda)K^n > 0$ for $u \in (t, \zeta)$, since $F > c(0)$. $\square$

In the following, we will consider only uncapped options.

2.4. European Options

By Proposition 4, non-negative values of L make it unprofitable to exercise a power call option prematurely, so it turns into a European-styled option. Non-dividend European power options have been studied by [25], while we consider the general case below.

Proposition 8. *The prices of the European power options are*

$$\begin{aligned}
P_{call} &= S_0^n e^{LT} N(d_1) - K^n e^{-(r+\lambda)T} N(d_2), \\
P_{put} &= K^n e^{-(r+\lambda)T} N(-d_2) - S_0^n e^{LT} N(-d_1),
\end{aligned}$$

where the constant L is defined by (4), and

$$\begin{aligned}
d_1 &= \frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2} + n\sigma^2 \right) T \right], \\
d_2 &= \frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2} \right) T \right].
\end{aligned}$$

Proof. Let $\overline{\mathbb{Q}}$ be the measure, defined by the Radon–Nikodym derivative

$$\zeta_t = \frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} \Big|_t = e^{-\frac{n^2\sigma^2}{2}t + n\sigma B_t}.$$

Obviously, the ζ -dynamics is $d\zeta_t = n\sigma dB_t$ and the $\overline{\mathbb{Q}}$ -Brownian motion is defined by

$$B_t^{\overline{\mathbb{Q}}} := -n\sigma t + B_t. \quad (8)$$

Hence, the call price can be derived as

$$P_{call} = e^{-(r+\lambda)T} \mathbb{E} \left[(S_T^n - K^n)^+ \right] = S_0^n e^{LT} \mathbb{E} [\zeta_T I_{S_T > K}] - K^n \mathbb{E} [I_{S_T > K}].$$

We finish the proof by changing the measure from $\mathbb{Q}$ to $\overline{\mathbb{Q}}$ and keeping in mind the form of the $\overline{\mathbb{Q}}$ -Brownian motion in (8). The put option can be evaluated analogously. $\square$

We shall prove a result that has an analogue for ordinary American options, namely that early exercise of an American call is never optimal when there are no dividends ($\lambda = 0$). First, we need the following lemma:

Lemma 1. *The process $e^{-(r+\lambda+L)t}S_t^n$ is a martingale, where L is given by (4).*

Proof. The lemma follows from the fact that $e^{-(r+\lambda+L)t}S_t^n = S_0^n \exp\left(-\frac{\sigma^2 n^2}{2}t + \sigma n B_t\right)$. $\square$

Proposition 9. *If $L \geq 0$, then early exercise is never optimal for an American power call option, making it European-styled.*

Proof. Suppose that the opposite is true, that is, there exists an optimal point (t, x) for a power call option with $L \geq 0$. Let ζ be an arbitrary stopping time with values between t and T . Using Lemma 1 and the inequalities $L \geq 0$ and $r + \lambda > 0$, we obtain

$$\begin{aligned}\mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\left(S_\zeta^n - K^n\right)^+\right] &\leq e^{-(r+\lambda)t}(x^n - K^n) \\ &= e^{Lt}\mathbb{E}^{t,x}\left[e^{-(r+\lambda+L)\zeta}S_\zeta^n\right] - K^n e^{-(r+\lambda)t} \\ &< \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\left(S_\zeta^n - K^n\right)\right] \\ &\leq \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\left(S_\zeta^n - K^n\right)^+\right].\end{aligned}$$

The contradiction completes the proof. $\square$

Remark 3. *Proposition 9 is consistent with the conclusions of Propositions 3 and 4.*

3. Perpetual Options

We now consider options without maturity constraints. The first results look at power call options.

Proposition 10 (Call). *If $L \geq 0$, then early exercise of a power call option is never optimal. Moreover, the option price is $P(x) = x^n$ when $L = 0$, and $P(x) = \infty$ when $L > 0$. If $L < 0$, then the early exercise boundary is given by*

$$c = K\left(\frac{\gamma_c}{\gamma_c - n}\right)^{\frac{1}{n}}, \quad (9)$$

and leads to the price

$$P(x) = x^{\gamma_c} K^{n-\gamma_c} \frac{n}{\gamma_c - n} \left(\frac{\gamma_c - n}{\gamma_c}\right)^{\frac{\gamma_c}{n}}, \quad (10)$$

when $x < c$. If $x \geq c$, then the price is $P(x) = x^n - K^n$.

Proof. Suppose that $L < 0$, the initial asset price is a small enough value x , and the option matures when the underlying asset hits the level $c > x$; we will denote this moment by ζ^c . Note that ζ^c can be viewed as the first hitting time of the level d_2 for a Brownian motion with drift $\mu = -d_1$, where d_1 and d_2 are given in (6). Also, γ_c in (6) can be rewritten as

$$\gamma_c = \frac{\sqrt{d_1^2 + 2(r + \lambda)} + d_1}{\sigma}.$$

Regarding the price, we have

$$\begin{aligned} P(x; c) &= \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^c} \left(S_{\zeta^c}^n - K^n \right)^+ I_{\zeta^c < \infty} \right] + x^n \lim_{T \rightarrow \infty} \mathbb{E}^x \left[e^{-(r+\lambda)T} (S_T^n - K^n)^+ I_{T < \zeta^c} \right] \\ &= (c^n - K^n) \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^c} I_{\zeta^c < \infty} \right] + x^n \lim_{T \rightarrow \infty} e^{\left(L - \frac{n^2 \sigma^2}{2}\right)T} \mathbb{E} \left[e^{n\sigma B_T} I_{T < \zeta^c, B_T > d_1 T + \frac{1}{\sigma} \ln \frac{K}{x}} \right] \\ &\quad - K^n \lim_{T \rightarrow \infty} e^{-(r+\lambda)T} \mathbb{Q} \left(T < \zeta^c, B_T > d_1 T + \frac{1}{\sigma} \ln \frac{K}{x} \right). \end{aligned} \quad (11)$$

Obviously, the last term tends to zero. From (A4), we see that the expectation in the second term tends to infinity as $e^{\left(\frac{n^2 \sigma^2}{2}\right)T}$. Hence, the second term also tends to zero, since this expectation is multiplied by $e^{\left(L - \frac{n^2 \sigma^2}{2}\right)T}$ and $L < 0$, see also Theorem 3.2 in [40]. Keeping (A8) in mind, we conclude that the price can be written as

$$P(x; c) = (c^n - K^n) \left(\frac{x}{c} \right)^{\gamma_c}, \quad (12)$$

and its derivative with respect to c is given by

$$\frac{\partial P(x; c)}{\partial c} = \frac{-(\gamma_c - n)c^n + K^n \gamma_c}{c^{\gamma_c + 1}} x^{\gamma_c}. \quad (13)$$

By Lemma A3, $\gamma_c > n$, so the price function (12) achieves its maximum for c given in (9). The formula (10) is obtained by plugging (9) into (12).

The case $L \geq 0$ was discussed in Proposition 9. In addition, Lemma A4 together with Propositions A1 and A6 shows that the second term in (11) is zero also in this case. Hence, the price is again given by (12). Here, however, the derivative (13) is always positive due to Lemma (A3); therefore, the optimal boundary must be $c = \infty$, confirming that early exercise of a call is never optimal when $L \geq 0$.

Suppose that $L > 0$. The option price can be calculated as the $c \rightarrow \infty$ limit of (11). By Lemma A3, $n > \gamma_c$, so the first term in (11), which is given by (12), tends to infinity. Since the second term is non-negative and the third one is zero, we conclude that $P(x) = \infty$.

The only case left to consider is $L = 0$ or, equivalently, $\gamma_c = n$. In this case, the first term in (11) tends to x^n . Using Proposition A1, we see that the second term becomes

$$\begin{aligned} &N\left(\frac{(d_1 - n\sigma)T + d_2}{\sqrt{T}}\right) - N\left(\frac{(d_1 - n\sigma)T + \frac{1}{\sigma} \ln \frac{K}{x}}{\sqrt{T}}\right) \\ &+ e^{2d_2(n\sigma - d_1)} \left(N\left(\frac{(d_1 - n\sigma)T + \frac{1}{\sigma} \ln \frac{K}{x} - 2d_2}{\sqrt{T}}\right) - N\left(\frac{(d_1 - n\sigma)T - d_2}{\sqrt{T}}\right) \right), \end{aligned} \quad (14)$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. By Lemma A4, its arguments tend to $-\infty$, as $T \rightarrow \infty$, because $d_1 - n\sigma < 0$. Therefore, (14) tends to zero. We complete the proof by observing that the third term in (11) is also zero. $\square$

We now turn to put-style options.

Proposition 11 (Put). *The early exercise boundary of a put option is*

$$c = K \left(\frac{\gamma_p}{\gamma_p + n} \right)^{\frac{1}{n}}, \quad (15)$$

which leads to the price

$$P(x) = \frac{K^{n+\gamma_p}}{x^{\gamma_p}} \frac{n}{\gamma_p + n} \left(\frac{\gamma_p}{\gamma_p + n} \right)^{\frac{\gamma_p}{n}}, \quad (16)$$

when $x > c$. If $x \leq c$, then $P(x) = K^n - x^n$.

Proof. Assume that the initial asset price is a sufficiently large enough x , and that the option matures at the first hitting of c , where $c < x$. The price function is given by

$$\begin{aligned} P(x; c) &= (K^n - c^n) \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^c} I_{\zeta^c < \infty} \right] \\ &\quad - x^n \lim_{T \rightarrow \infty} e^{\left(L - \frac{n^2 \sigma^2}{2}\right)T} \mathbb{E} \left[e^{n\sigma B_T} I_{T < \zeta^c, B_T < d_1 T + \frac{1}{\sigma} \ln \frac{K}{x}} \right] \\ &\quad + K^n \lim_{T \rightarrow \infty} e^{-(r+\lambda)T} \mathbb{Q} \left(T < \zeta^c, B_T < d_1 T + \frac{1}{\sigma} \ln \frac{K}{x} \right). \end{aligned} \quad (17)$$

Similar arguments to the ones in the proof of Proposition 10 show that the second and the third term in (17) are zero whether $L < 0$ or $L \geq 0$. In this case, however, we have to use (A3) instead of (A4). The first term can be evaluated using (A9), so that (17) becomes

$$P(x; c) = (K^n - c^n) \left(\frac{c}{x} \right)^{\gamma_p}, \quad (18)$$

and its derivative with respect to c is given by

$$\frac{\partial P(x; c)}{\partial c} = [-(\gamma_p + n)c^n + K^n \gamma_p] \frac{c^{\gamma_p - 1}}{x^{\gamma_p}}.$$

Thus, we can easily check that the price (18) achieves its maximum for the boundary given by (15), which leads to the price (16) when $x > c$. $\square$

Corollary 2. The optimal boundary has the following asymptotic behavior as a function of n .

1. The call optimal boundary tends to Ke^{1/γ_c} , as $n \rightarrow 0$.
2. The put optimal boundary tends to Ke^{-1/γ_p} , as $n \rightarrow 0$, and to the strike, as $n \rightarrow \infty$.

Proof. The corollary is a consequence of (9) and (15). $\square$

4. Finite Maturities

In this section, we consider options with finite maturities and try to approximate the entire optimal boundary. When working on a denser grid, the free boundary equation that describes the option price turns into a boundary value problem in a known region, and we present a Crank–Nicolson finite difference approach to solve it.

4.1. Optimal Boundary

Let us divide the time interval into l sub-intervals. Thus, we shall evaluate the optimal boundary at the time grid $0 \equiv t_0 < t_1 < \dots < t_l \equiv T$ using exponents of continuous piecewise linear functions, $\exp(a_i t + b_i) I_{t \in [t_{i-1}, t_i]}$. We denote the values at the grid nodes by $c_i = \exp(a_i t_i + b_i) = \exp(a_{i+1} t_i + b_{i+1})$. Let x be the initial value of the underlying asset, and ζ the first hitting time of the boundary. For a call option, we exclude the case $L \geq 0$, where the optimal boundary is infinitely large, and assume that x is small enough. Conversely, for a put option, we assume a large enough value for x . We can view the stopping time ζ as the first hitting time for a Brownian motion of the function,

$$\frac{1}{\sigma} \left(\left(a_i - r + \frac{\sigma^2}{2} \right) t + b_i - \log(x) \right) = C_i t_i + D_i,$$

where

$$C_i = \frac{1}{\sigma} \left(a_i - r + \frac{\sigma^2}{2} \right) \quad \text{and} \quad D_i = \frac{b_i - \log(x)}{\sigma},$$

if the hitting happens in the interval $[t_{i-1}, t_i)$. Under these assumptions, the call price function is given by

$$\begin{aligned} P(x; \{t_0, \dots, t_l\}; \{c_0, \dots, c_l\}) &= \mathbb{E}^x \left[e^{-(r+\lambda)(\zeta \wedge T)} \left(S_{\zeta \wedge T}^n - K^n \right)^+ \right] \\ &= \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} \left(S_{\zeta}^n - K^n \right) I_{\zeta \leq T} \right] + e^{-(r+\lambda)T} \mathbb{E}^x \left[\left(S_T^n - K^n \right)^+ I_{\zeta > T} \right] \\ &= x^n \sum_{m=1}^l e^{n\sigma D_m} \mathbb{E} \left[e^{-\alpha_{2,m}\zeta} I_{t_{m-1} < \tau \leq t_m} \right] - K^n \mathbb{E} \left[e^{-\alpha_1 \tau} I_{\zeta \leq T} \right] \\ &\quad + x^n e^{-\alpha_3 T} \mathbb{E} \left[e^{n\sigma B_T} I_{B_T > k, \zeta > T} \right] - K^n e^{-\alpha_1 T} \mathbb{Q}(B_T > k, \zeta > T), \end{aligned} \quad (19)$$

where

$$\begin{aligned} \alpha_1 &= r + \lambda, \\ \alpha_{2,m} &= (r + \lambda) - n \left(r - \frac{\sigma^2}{2} \right) - n C_m \sigma = n \frac{\sigma^2}{2} - (n-1)r - n C_m \sigma + \lambda, \\ \alpha_3 &= \lambda + n \frac{\sigma^2}{2} - (n-1)r, \\ k &= \frac{1}{\sigma} \log \left(\frac{K}{x} \right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2} \right) T, \end{aligned}$$

and the put price function is given by

$$\begin{aligned} P(x; \{t_0, \dots, t_l\}; \{c_0, \dots, c_l\}) &= \mathbb{E}^x \left[e^{-(r+\lambda)(\zeta \wedge T)} \left(K^n - S_{\zeta \wedge T}^n \right)^+ \right] \\ &= K^n \mathbb{E} \left[e^{-\alpha_1 \tau} I_{\zeta \leq T} \right] - x^n \sum_{m=1}^l e^{n\sigma D_m} \mathbb{E} \left[e^{-\alpha_{2,m}\zeta} I_{t_{m-1} < \tau \leq t_m} \right] \\ &\quad + K^n e^{-\alpha_1 T} \mathbb{Q}(B_T > k, \zeta > T) - x^n e^{-\alpha_3 T} \mathbb{E} \left[e^{n\sigma B_T} I_{B_T < k, \zeta > T} \right]. \end{aligned} \quad (20)$$

The expectations in (20) can be obtained from Propositions A4 and A5. On the other hand, call pricing leads to an upper hitting problem, so the expectations in (19) can be derived by symmetric arguments, see also [41].

We approximate the optimal boundary backwards. Let us first consider a put option. The level c_l corresponds to the level at maturity, so its value can be obtained from Proposition 5. Suppose we know all the values after some moment m , i.e., $c_m, c_{m+1}, \dots, c_l$. Consider the price $h(c) := P(x; \{0, t_m - t_{m-1}, \dots, t_l - t_{m-1}\}; \{c, c_m, \dots, c_l\})$, defined in (20), as a function of c in the interval $(0, c_m)$. We choose this interval because the optimal boundary decreases with respect to the time to maturity. Let $c(x)$ be the value of c for which the function $h(c)$ achieves its maximum for a given initial point x . Then our approximation for the optimal boundary at the previous grid node will be the largest x for which $c(x) = x$ and corresponds to the largest initial point at which immediate exercise is optimal.

The optimal boundary for a call option is evaluated in a similar way, with c_l given in Proposition 4. Note that we only need to consider the case $L < 0$. Now, the function $h(c)$ is given by (19), so we maximize it in the interval (c_m, ∞) , since the call boundary increases with respect to the time to maturity. Our approximation is the smallest x for which $c(x) = x$ and corresponds to the lowest initial point for which immediate exercise is optimal.

Once we approximate the optimal boundary, we can use Formulas (19)–(20) to derive the option prices quickly and with relatively high accuracy.

4.2. Pricing as a Boundary Value Problem

Alternatively, we may approximate the optimal boundary at a denser grid and consider option pricing as a boundary value problem (BVP, hereafter) in a known region. For a call option, the region is $\{(0, T) \times (0, c(t))\}$ and leads to the BVP

$$\begin{aligned} F_t(t, x) + rxF_x(t, x) + \frac{1}{2}\sigma^2x^2F_{xx}(t, x) - rF(t, x) &= 0, \\ F(t, 0) &= 0, \quad \text{for } t \in (0, T), \\ F(t, c(t)) &= e^{-\lambda t}(x^n - K^n), \quad \text{for } t \in (0, T), \\ F(T, x) &= e^{-\lambda T}(x^n - K^n)^+, \quad \text{for } x \in (0, c_l). \end{aligned} \quad (21)$$

For a put option, the region is $\{(0, T) \times (c(t), \infty)\}$ and leads to the BVP

$$\begin{aligned} F_t(t, x) + rxF_x(t, x) + \frac{1}{2}\sigma^2x^2F_{xx}(t, x) - rF(t, x) &= 0, \\ F(t, \infty) &= 0, \\ F(t, c(t)) &= e^{-\lambda t}(K^n - x^n), \quad \text{for } t \in (0, T), \\ F(T, x) &= e^{-\lambda T}(K^n - x^n)^+, \quad \text{for } x \in (c_l, \infty). \end{aligned} \quad (22)$$

Note that the continuation region is open from above. It is therefore useful to introduce a sufficiently large enough upper bound $\bar{C}$ where the price function is approximately zero.

4.3. Crank–Nicolson Finite Difference Approach

Equations (21) and (22), which describe the options' valuation problem, are modifications of the heat equation, which is widely studied in mathematics and physics, and there are several classical finite difference methods for solving it. The explicit algorithm is relatively fast, but it is stable only when the ratio between the time step and the square of the state step is below 0.5. Furthermore, the accuracy is proportional to the time step and the square of the state one. On the other hand, the implicit algorithm has the same accuracy but is always convergent. The cost of this feature is the relatively larger computational time due to the emergent linear systems that must be solved at each step. Another widely used method is the Crank–Nicolson approach. The error of this method is significantly smaller, since it is also proportional to the square of the time step. Despite the fact that this approach is only A-stable and not L-stable, its speed and accuracy motivate its use. Moreover, this method is specifically constructed for heat-style equations, which explains its wide application in the field. More precise results in terms of stability and accuracy can be derived from the theory of the classical heat equation, $f_t = f_{xx}$, which in our case is obtained from

$$F(t, x) = x^{-\alpha} e^{-\beta(T-t)} f(A(T-t), \ln(x)),$$

where

$$A = \frac{\sigma^2}{2}, \quad B = r - \frac{\sigma^2}{2}, \quad \alpha = \frac{B}{2A} = \frac{r}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{B^2}{4A} + r = \frac{1}{2} \left(\frac{r}{\sigma^2} + \frac{1}{2} \right)^2.$$

We present a method for solving the above BVP in a closed region based on the Crank–Nicolson finite difference approach. Let us assume that the lower boundary is $c_1(\cdot)$ and the upper boundary is $c_2(\cdot)$.

1. We divide the space $[0, T] \times [c_1(T), c_2(T)]$ uniformly with M time nodes and N state nodes, $T \equiv t_1 > t_2 > \dots > t_M \equiv 0$ and $c_1(T) \equiv x_1 < x_2 < \dots < x_N \equiv c_2(T)$. We denote by $F(i, j)$ the solution at the (i, j) -th node, where i corresponds to time and j to the state variable. The values $F(1, j)$ are the prices at maturity, while $F(M, j)$ are the initial option prices; note that we work backwards.
2. We approximate the boundary $c(t)$ at $\bar{M}$ points using the algorithm from Section 4.1. For a call option, $c(t)$ corresponds to the upper boundary $c_2(t)$, while $c_1(t) = 0$. For a put option, $c_1(t) = c(t)$ and $c_2(t) = \bar{C}$. Note that we choose $\bar{M}$ to be significantly lower than M in order to reduce computational time. We then estimate the entire boundary using a cubic spline interpolation.
3. The terminal condition is incorporated by

$$F(1, j) = e^{-\lambda T} (x_j^n - K^n)^+ \quad \text{for call options,}$$

$$F(1, j) = e^{-\lambda T} (K^n - x_j^n)^+ \quad \text{for put options.}$$

4. Let us denote by C_i^1 and C_i^2 the boundaries' values at the grid nodes. Let l_i be the largest j such that $x_{l_i} < C_i^1$, and k_i the smallest j such that $x_{k_i} > C_i^2$. Then $l_i = 1$ for call options, and $k_i = N$ for put options.
5. For a call option, the lower and upper boundary conditions are incorporated by

$$F(i, 1) = 0,$$

$$F(i, j) = e^{-\lambda t_i} (x_j^n - K^n)^+, \quad \text{for all } i \text{ and } j \geq k_i.$$

For a put option,

$$F(i, j) = e^{-\lambda t_i} (K^n - x_j^n)^+, \quad \text{for all } i \text{ and } j \leq l_i,$$

$$F(i, N) = 0.$$

6. We derive the values of $F(i, j)$ iteratively using the already obtained $F(q, j)$ for all j and $q < i$ by the Crank–Nicolson scheme. The corresponding derivatives are approximated by (A10), and the BVP (21) or (22) can be represented as (A11). Rearranging in terms of $j \in \{l_i + 1, \dots, k_i - 1\}$, we arrive at the linear system (A12)–(A14) for $F(i, j)$.

5. Numerical Results

We present several numerical experiments based on the following parameter values: risk-free rate $r = -0.03$, additional discount factor $\lambda = 0.07$, volatility $\sigma = 0.3$, strike $K = \$20$, and power n between zero and two. We assume that the initial asset price is $S_0 = \$30$ for call options, and $S_0 = \$10$ for put options.

Figure 1a,b compare the optimal boundaries for different powers: $n = 0.5$, $n = 1$, $n = 1.5$, and $n = 2$. The results are given with respect to time t , assuming a maturity of $T = 100$. The first figure relates to call options, while the second figure relates to put options, with perpetual values marked by circles. The values of the constant L , defined in (4), are $L = -0.0663$, $L = -0.0700$, $L = -0.0513$, and $L = -0.0100$, respectively. It turns out that L changes its sign at $\tilde{n} = 2.0916$.

The entire boundary surface can be seen in Figure 1c,d. Note that here we parametrize with respect to the time to maturity. It follows that negative values of L lead to a finite call boundary, which is larger when n increases, and tends to infinity when n tends to the critical value at which L changes its sign, $\tilde{n} = 2.0916$. As shown in Corollary 1 and Proposition 6, the initial boundary detaches from the strike at the point $\bar{n} = 1 - 2r/\sigma^2 = 1.6667$, marked

by a red asterisk. Figure 1e,f shows the behavior of the price, with red circles denoting the corresponding perpetual values.

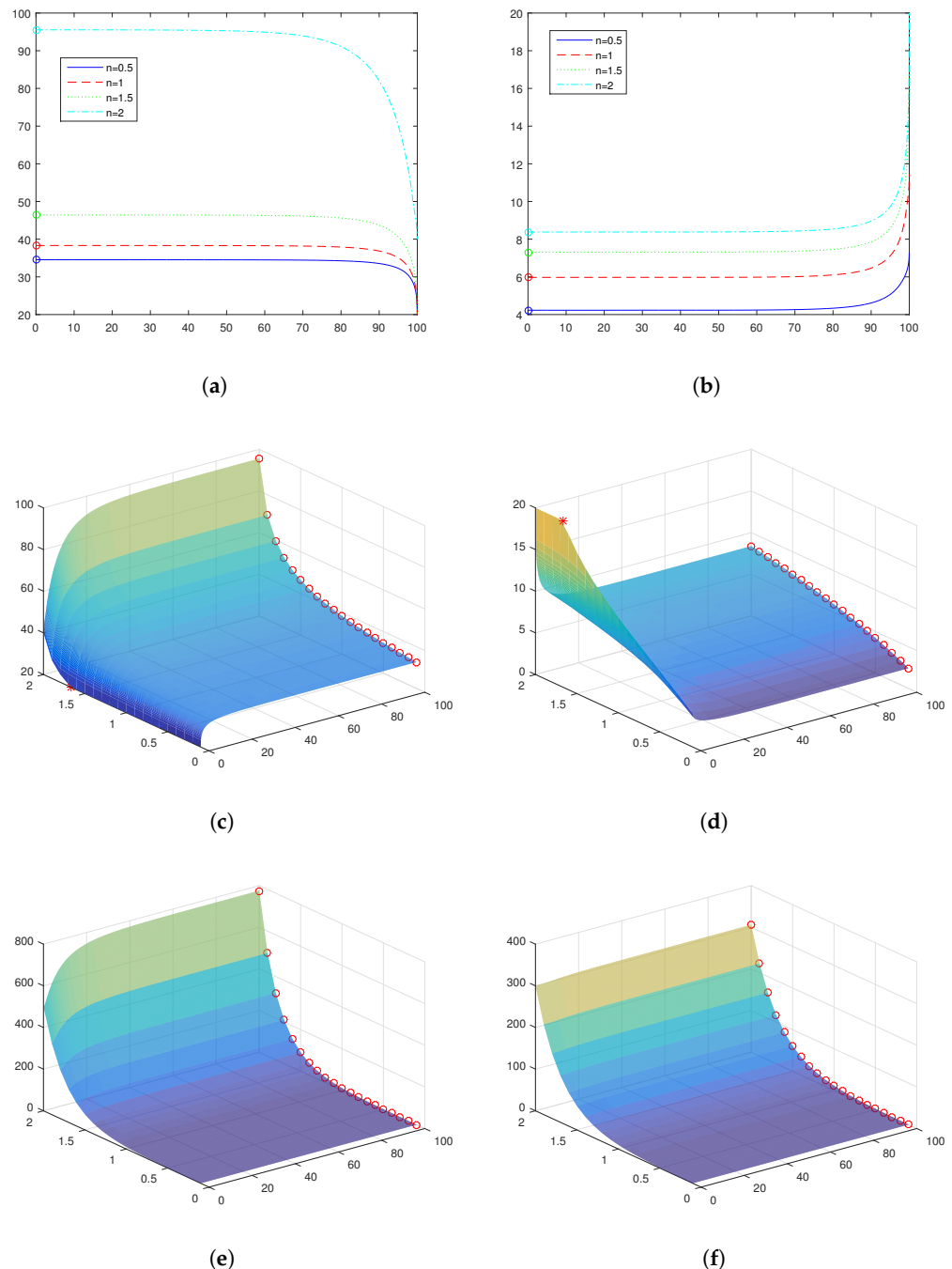


Figure 1. Optimal boundaries and prices. (a) Call boundaries; (b) put boundaries; (c) call boundary surface; (d) put boundary surface, the red asterisk marks the point at which the initial boundary detaches from the strike; (e) call prices; (f) put prices.

The behavior of the initial and perpetual values of the put boundary is given as a function of the power in Figure 2a. Formula (7) applied to the current parameters implies an initial value of 3.0671, as $n \rightarrow 0$, which is marked by a blue point on the graph. We denote with a blue asterisk the level after which the initial boundary coincides with the strike, which occurs at $\bar{n} = 1.6667$. By Corollary 2, as $n \rightarrow 0$, the perpetual value becomes $Ke^{-1/\gamma_p} = 1.9015$, which is plotted by a red circle. The same corollary shows that the limit, as $n \rightarrow \infty$, is the strike.

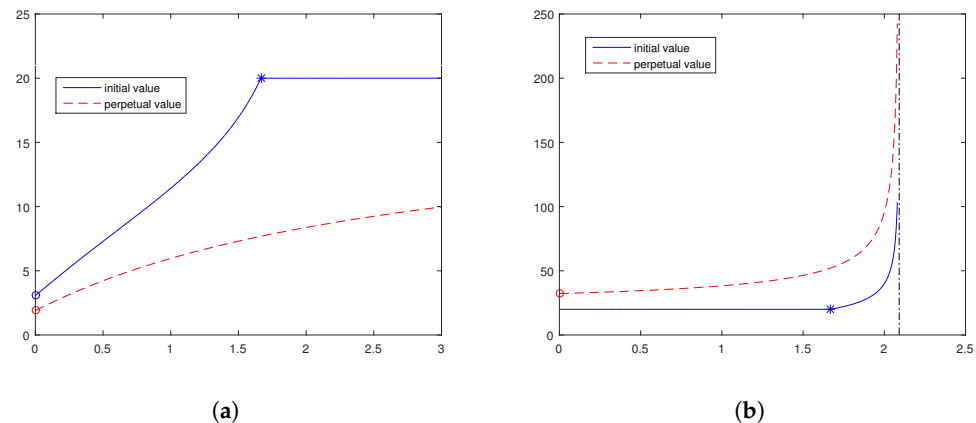


Figure 2. Optimal boundaries w.r.t the power. (a) Put; (b) call. The levels below which the initial boundary coincides with the strike are marked by blue asterisks.

The behavior of the call boundaries can be seen in Figure 2b. The level below which the initial boundary coincides with the strike is $\bar{n} = 1.6667$ and is marked by a blue asterisk. By Corollary 2, as $n \rightarrow 0$, the perpetual value becomes $Ke^{1/\gamma_c} = 32.2599$. The level at which L changes its sign is $\tilde{n} = 2.0916$ and is indicated by the black dash-dotted line. If the power is larger than this value, $n \geq \tilde{n}$, then early exercise is never optimal, which means that the boundaries are infinitely large. We can also see that both boundaries tend to infinity, as $n \rightarrow \tilde{n}^-$.

Finally, we compare the speed of our approach with the classical approach by Cox et al. [42] based on binomial trees. The computations were performed with the MATLAB 2024a software platform on a 13th Gen Intel(R) Core(TM) i7-1355U 1.70 GHz computer configuration. Put option prices and time consumption are given in Table 1. The parameters used are $r = -0.01$, $\lambda = 0.03$, $\sigma = 0.3$, $K = 20$, $S_0 = 20$, and $T = 3$. The comparison is made for classical American options (i.e., $n = 1$), since the binomial tree method is implementable specifically in this case. We execute our algorithm with one, two, three, and four steps. On the other hand, we use binomial trees with time lags of 0.1, 0.005, 0.0005, and 0.0001. We can conclude that our approach is significantly faster with respect to the required accuracy.

Table 1. American option prices and the computational times.

our approach	one step	two steps	three steps	four steps
price	4.1081	4.1141	4.1147	4.1148
time in seconds	0.008572	0.032991	0.163783	1.615244
binomial trees	grid length	grid length	grid length	grid length
	0.1	0.005	0.0005	0.0001
price	4.0836	4.1132	4.1147	4.1148
time	0.008564	0.079593	4.630226	161.668976

6. Some Evidence for the S&P 500 Index

We will investigate the behavior of the financial markets in the wake of the 2008 global financial crisis, which began on Sep 15, 2008 with the default of Lehman Brothers. A series of turbulent days followed, which led to outstanding securities falling by nearly ten percent on 29 September, from USD 1213.27 to USD 1106.42, and the specific reason was the rejection of the Emergency Economic Stabilization Act in the U.S. Congress. For our purpose, we will use the prices of several European-style options on that date as well as on the previous business day, 26 September 2008. These options mature on 20 December

2008, and the risk-free rate on 26 and 29 September is 0.83% and 0.45%, respectively, which are calculated using 13-week Treasury bills. The data set contains 27 call and 31 put options with different strikes for 26 September 2008, and 45 call and 37 put options for 29 September 2008. More precisely, we use the best bid and ask values, and obtain the price by averaging. From these prices, we derive the implied volatilities that characterize the options and apply the approach outlined in this paper to estimate the associated European and American power options. In this way, we can compare the state of the market before and after the shock on 29 September 2008.

The strips for the call and put implied volatilities generated by the ask-bid spread can be seen in Figure 3a,b. The values calculated at the midpoints are given by the solid lines. We can draw several conclusions: first, both ask-bid spreads lead to a relatively narrow strip before the market shock. Also, both the call and put strips have joint points. On the contrary, the ask-bid spreads are very large on 29 September 2008, leading to a very wide strip for the implied volatility. Moreover, the put and call strips are very far apart and do not intersect. Also, the best bid for the deeply out-the-money put option (strike \$1250) is \$139.50, which is outside the admissible range for put options, $(\max\{Ke^{-rT} - S_0, 0\}, Ke^{-rT})$. This would result in zero volatility; see the put strip in Figure 3b.

The prices of hypothetical American-style power options on 26 and 29 September 2008 can be seen in Figure 3c,d for the calls, and Figure 3e,f for the puts. Note that the CBOE-listed options written on the S&P 500 Index (SPX) are European-styled only. On the other hand, the SPDR S&P 500 ETF Trust is a fund that tries to replicate the index. The options written on it, the so-called SPY options, are American-styled. In this study, we use only statistical data for the S&P 500 Index.) on 26 and 29 September 2008 can be seen in Figure 3c,d for the calls, and Figure 3e,f for the puts. The graphs are given with respect to the strike and (implied) volatility. We have plotted three graphs with respect to different powers: $n = 0.9$, $n = 1$, and $n = 1.1$, so that the middle graph corresponds to classical American options. Red circles mark the observed real market prices of European options. Note that they are very close to the prices of the corresponding American options because of the short time to maturity and low interest rate. Moreover, American options for $n = 1$ and $n = 1.1$ become European options, by Proposition 9.

The prices of some options with strike \$1200 are given in Table 2. Columns 2 to 4 report the option style (call or put), maturity date, and the observed market price, respectively. The next column contains the implied volatilities that these prices generate. We then give the prices of European and American power options for powers $n \in \{0.8; 1; 1.2\}$, which are calculated with respect to the implied volatilities. Note that the prices of both European and American call options coincide for $n = 1$ or $n = 1.1$, by Proposition 9. Also, these prices are equal to the observed ones when $n = 1$.

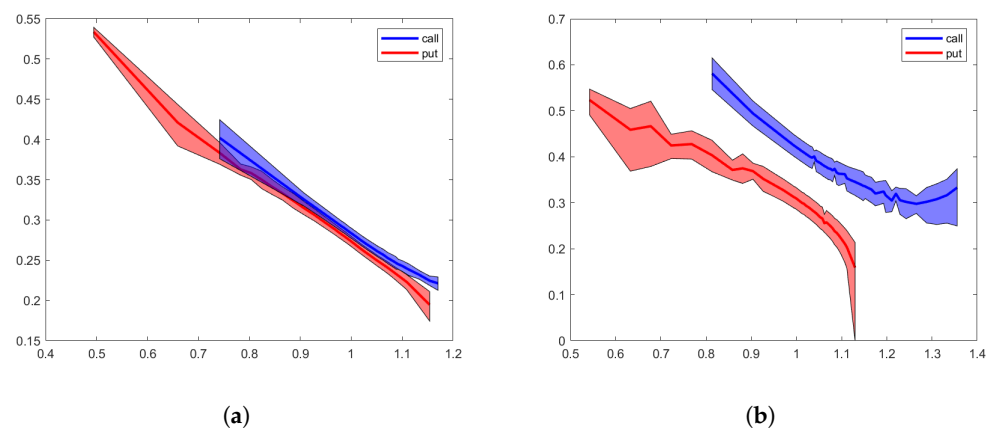


Figure 3. Cont.

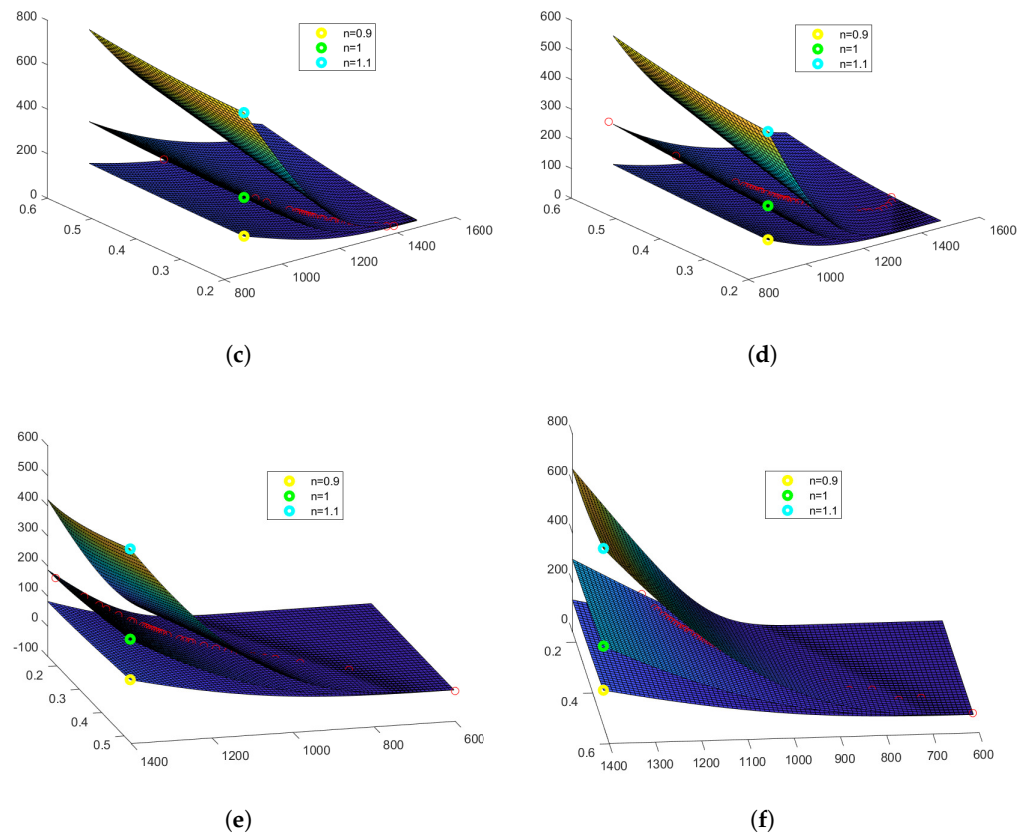


Figure 3. Implied volatilities, short maturities—20 December 2008: (a) 26 September 2008; (b) 29 September 2008; (c) 26 September, call; (d) 29 September, call; (e) 26 September, put; (f) 29 September, put.

Table 2. Option prices.

Strike	C/P	Maturity	Real	I.V.	$n = 0.8$ AM	$n = 0.8$ EU	$n = 1$ AM	$n = 1$ EU	$n = 1.2$ AM	$n = 1.2$ EU
26 September 2008, $S_0 = 1213.27$										
1200	C	30.09.2008	30.85	0.5498	5.9395	5.9374	30.85	30.85	153.8870	153.8870
1200	C	03.10.2008	35.90	0.4228	6.9023	6.8999	35.90	35.90	179.3238	179.3238
1200	C	18.10.2008	50.05	0.3526	9.5866	9.5829	50.05	50.05	250.9715	250.9715
1200	C	22.11.2008	66.60	0.3035	12.6989	12.6950	66.60	66.60	335.4794	335.4794
1200	C	20.12.2008	75.80	0.2863	14.4165	14.4129	75.80	75.80	382.7927	382.7927
1200	C	17.01.2009	83.50	0.2827	15.8482	15.8437	83.50	83.50	422.5867	422.5867
1200	C	21.03.2009	98.70	0.2689	18.6552	18.6506	98.70	98.70	501.6477	501.6477
1200	C	20.06.2009	116.70	0.2608	21.9497	21.9430	116.70	116.70	596.1868	596.1868
1200	C	19.09.2009	132.20	0.2542	24.7599	24.7510	132.20	132.20	678.4027	678.4027
1200	C	19.12.2009	145.70	0.2500	27.1877	27.1758	145.70	145.70	750.6416	750.6416
1200	C	18.12.2010	195.10	0.2503	35.9213	35.8747	195.10	195.10	1020.4562	1020.4562
1200	P	30.09.2008	15.60	0.5043	3.0381	3.0381	15.6013	15.60	76.9366	76.8992
1200	P	03.10.2008	21.10	0.4028	4.1152	4.1152	21.1045	21.10	103.9476	103.8625
1200	P	18.10.2008	33.60	0.3314	6.5742	6.5743	33.6209	33.60	165.1373	164.8645
1200	P	22.11.2008	49.70	0.2931	9.7657	9.7648	49.7683	49.70	243.6060	242.8737
1200	P	20.12.2008	57.80	0.2764	11.3833	11.3799	57.9118	57.80	282.9874	281.8824
1200	P	17.01.2009	64.80	0.2737	12.7863	12.7808	64.9492	64.80	316.9138	315.4716
1200	P	21.03.2009	78.70	0.2630	15.5913	15.5776	78.9611	78.70	384.1912	381.8228
1200	P	20.06.2009	94.70	0.2234	18.8845	18.8254	95.3293	94.70	462.3773	457.5418
1200	P	19.12.2009	118.30	0.2463	23.7229	23.6543	119.1421	118.30	575.1595	568.3797
1200	P	18.12.2010	149.30	0.2351	30.2710	30.0954	151.0575	149.30	724.8998	711.8362

Table 2. Cont.

Strike	C/P	Maturity	Real	I.V.	$n = 0.8$ AM	$n = 0.8$ EU	$n = 1$ AM	$n = 1$ EU	$n = 1.2$ AM	$n = 1.2$ EU
29 September 2008, $S_0 = 1213.27$										
1200	C	03.10.2008	6.90	0.6182	1.3281	1.3275	6.90	6.90	34.4304	34.4304
1200	C	18.10.2008	20.00	0.4851	3.8318	3.8286	20.00	20.00	100.3090	100.3090
1200	C	22.11.2008	35.00	0.3940	6.6744	6.6666	35.00	35.00	176.4388	176.4388
1200	C	20.12.2008	44.90	0.3707	8.5373	8.5253	44.90	44.90	227.0824	227.0824
1200	C	17.01.2009	50.45	0.3531	9.5768	9.5624	50.45	50.45	255.6114	255.6114
1200	C	21.03.2009	64.00	0.3262	12.1010	12.0794	64.00	64.00	325.6877	325.6877
1200	C	20.06.2009	104.10	0.3731	19.4701	19.4000	104.10	104.10	536.8327	536.8327
1200	C	19.12.2009	106.30	0.2876	19.8560	19.8000	106.30	106.30	548.4694	548.4694
1200	C	18.12.2010	201.80	0.3570	36.7313	36.4060	201.80	201.80	1077.6160	1077.6160
1200	P	22.11.2008	102.30	0.2931	20.1159	20.1140	102.4501	102.30	501.1520	499.5529
1200	P	20.12.2008	111.40	0.2764	21.9729	21.9726	111.5755	111.40	544.3233	542.3102
1200	P	31.12.2008	114.30	0.2764	22.5652	22.5652	114.4770	114.30	558.0388	555.9327
1200	P	17.01.2009	117.30	0.2737	23.1797	23.1795	117.5039	117.30	572.3521	569.9970
1200	P	21.03.2009	129.60	0.2630	25.7060	25.7043	129.8921	129.60	630.7909	627.5267
1200	P	20.06.2009	144.30	0.2234	28.7438	28.7391	144.7150	144.30	700.4051	695.9071
1200	P	19.12.2009	166.60	0.2463	33.4016	33.3838	167.2934	166.60	805.7800	798.7695

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Appendix A. Some Propositions

Lemma A1. If $G(x) = (K^n - x^n)$ and $BG(x) < 0$ for some x , then $BG(y) < 0$ for all $y < x$.

Proof. We can easily check that $BG(x) = -Lx^n - (r + \lambda)K^n$, where L is given by (4). $\square$

Lemma A2. Let the constant L be defined as in (4).

1. If $n < 1$, then $L < 0$.
2. If $L < 0$ and $r + (n - 1)\sigma^2/2 > 0$, then $-(r + \lambda)/L > 1$.

Proof. Regarding the first statement, if $r \leq 0$, we can rewrite (4) as $L = (n - 1)n\sigma^2/2 + nr - (r + \lambda)$ to see that $L < 0$; if $r > 0$, then we can use (4) directly.

The second statement is reached by several easy calculations. $\square$

Lemma A3. Let L , γ_c , and γ_p be defined as in (4) and (6). If $L = 0$, then $\gamma_c = n$ and $\gamma_p = n + 2\frac{r}{\sigma^2} - 1$. Otherwise, if $L \neq 0$, then L , $n - \gamma_c$, and $n + 2\frac{r}{\sigma^2} - 1 - \gamma_p$ have the same sign.

Proof. We can rewrite γ_c and γ_p as

$$\gamma_{p,c} = \sqrt{\left(n + \frac{r}{\sigma^2} - \frac{1}{2}\right)^2 - 2\frac{L}{\sigma^2}} \pm \left(\frac{r}{\sigma^2} - \frac{1}{2}\right).$$

It follows from (4) that $r + \lambda > 0$ is equivalent to

$$\frac{n}{2} + \frac{r}{\sigma^2} - \frac{1}{2} > \frac{L}{n\sigma^2}. \quad (\text{A1})$$

If $L \geq 0$, we obtain the stated result, since (A1) leads to

$$n + \frac{r}{\sigma^2} - \frac{1}{2} > 0. \quad (\text{A2})$$

If $L < 0$, we use $\sqrt{x^2} \geq x$ to complete the proof. $\square$

Lemma A4. Let L and d_1 be defined as in (4) and (6). Then $2L < (d_1 - n\sigma)^2$. Moreover, if $L \geq 0$, then $d_1 - n\sigma < 0$.

Proof. The first statement follows from

$$(d_1 - n\sigma)^2 - 2L = \left(\frac{\sigma}{2} - \frac{r}{\sigma}\right)^2 + 2(r + \lambda).$$

The inequality $d_1 - n\sigma < 0$ is equivalent to (A2), which is true when $L \geq 0$. $\square$

Appendix B. Some Hitting Time Properties

Let $c(t)$ be a continuous piecewise linear function, $c(t) = (a_mt + b_m)I_{t_{m-1} < t \leq t_m}$, with respect to the time grid $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$, and $c_m = a_mt_m + b_m$. We assume that $c(0) < 0$, i.e., a lower hitting problem. The upper one is analogous, since Brownian motion is symmetric. Let us denote by ζ the first hitting time of the boundary. We first consider linear functions and write $c(t) = at + b$.

Proposition A1. If $b = c(0) < 0$ and $z > c(T)$, then

$$\begin{aligned} V(\alpha, z, T) &\equiv \mathbb{E}\left[e^{\alpha B_T} I_{B_T < z, T < \zeta}\right] \\ &= \exp\left(\frac{T\alpha^2}{2}\right) \left[\frac{N\left(\frac{z - T\alpha}{\sqrt{T}}\right) - N\left(\frac{c(T) - T\alpha}{\sqrt{T}}\right)}{-e^{2b(\alpha - a)} \left(N\left(\frac{z - T\alpha - 2b}{\sqrt{T}}\right) - N\left(\frac{c(T) - T\alpha - 2b}{\sqrt{T}}\right)\right)} \right]. \end{aligned} \quad (\text{A3})$$

If $b = c(0) > 0$ and $z < c(T)$, then

$$\begin{aligned} V(\alpha, z, T) &\equiv \mathbb{E}\left[e^{\alpha B_T} I_{B_T > z, T < \zeta}\right] \\ &= \exp\left(\frac{T\alpha^2}{2}\right) \left[\frac{N\left(\frac{c(T) - T\alpha}{\sqrt{T}}\right) - N\left(\frac{z - T\alpha}{\sqrt{T}}\right)}{+e^{2b(\alpha - a)} \left(N\left(\frac{z - T\alpha - 2b}{\sqrt{T}}\right) - N\left(\frac{c(T) - T\alpha - 2b}{\sqrt{T}}\right)\right)} \right]. \end{aligned} \quad (\text{A4})$$

Proof. The proof is a consequence of Theorem 3.2 in [41]. $\square$

Proposition A2. The probability of $\zeta < T$ is

$$g(T; a, b) \equiv \mathbb{P}(\zeta < T) = N\left(\frac{aT + b}{\sqrt{T}}\right) + \exp(-2ab)N\left(\frac{-aT + b}{\sqrt{T}}\right). \quad (\text{A5})$$

Proof. See Equation (3) in [43] or Proposition 3.1 in [41]. $\square$

Proposition A3. If $\alpha > 0$, then

$$L(T, \alpha; a, b) \equiv \mathbb{E}[e^{-\alpha\tau} I_{\zeta < T}] = e^{-b(\sqrt{a^2 + 2\alpha} + a)} g\left(T; -\sqrt{a^2 + 2\alpha}, b\right), \quad (\text{A6})$$

where the function $g(\cdot)$ is given by (A5).

Proof. See Theorem 3.1 in [41]. $\square$

We now consider piecewise linear boundaries.

Proposition A4. If $\alpha > 0$, then

$$\mathbb{E}\left[e^{-\alpha\tau} I_{\tau \in (t_{m-1}, t_m)}\right] = \int_{c_1, \dots, c_{m-1}}^{\infty} \left(\frac{\prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(c_{i-1} - x_{i-1})(c_i - x_i)}{t_i - t_{i-1}}\right)\right)}{\prod_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}}}\right) dx_1 \dots dx_{m-1},$$

where the function $L(\cdot)$ is given by (A6).

Proof. See Theorem 4.1 in [41]. $\square$

Corollary A1. If $z > c(T)$, then

$$U(z, T; c(\cdot)) \equiv \mathbb{P}(B_T < z, I_{T < \zeta}) = N\left(\frac{z}{\sqrt{T}}\right) - N\left(\frac{c(T)}{\sqrt{T}}\right) - e^{-2ba} \left[N\left(\frac{z - 2b}{\sqrt{T}}\right) - N\left(\frac{c(T) - 2b}{\sqrt{T}}\right) \right]. \quad (\text{A7})$$

Proof. We have to rewrite (A3) for $\alpha = 0$. $\square$

Proposition A5. Formulas (A3) and (A7) for piecewise linear boundaries become

$$\begin{aligned} & \mathbb{E}\left[e^{\alpha B_T} I_{B_T < z, \Phi_T = 1}\right] \\ &= \int_{c_1, \dots, c_{n-1}}^{\infty} \left(\frac{\prod_{i=1}^{n-1} \left(1 - \exp\left(-\frac{2(c_{i-1} - x_{i-1})(c_i - x_i)}{t_i - t_{i-1}}\right)\right)}{\prod_{i=1}^{n-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}}}\right) dx_1 \dots dx_{n-1}, \end{aligned}$$

and

$$\mathbb{P}(B_T < z, \Phi_T = 1) = \int_{c_1, \dots, c_{n-1}}^{\infty} \left(\frac{\prod_{i=1}^{n-1} \left(1 - \exp\left(-\frac{2(c_{i-1}-x_{i-1})(c_i-x_i)}{t_i-t_{i-1}}\right) \right)}{\prod_{i=1}^{n-1} \frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}}}}{e^{\alpha x_{n-1}} U(z - x_{n-1}, t_n - t_{n-1}; c(\cdot) - x_{n-1})} \right) dx_1 \dots dx_{n-1},$$

where the functions $V(\cdot)$ and $U(\cdot)$ are given by (A3) and (A7).

Proof. See Theorem 4.2 in [41]. $\square$

Proposition A6. The following statements hold for any constants $k > 0$ and $m < 0$.

1. If $k > m^2/2$, then $\lim_{T \rightarrow \infty} e^{kT} N(m\sqrt{T}) = \infty$.
2. If $k \leq m^2/2$, then $\lim_{T \rightarrow \infty} e^{kT} N(m\sqrt{T}) = 0$.

Proof. The proof follows from $\frac{x}{x^2+1} \leq e^{\frac{x^2}{2}} \int_x^{\infty} e^{-\frac{u^2}{2}} du \leq \frac{1}{x}$, which is proven in [44]. $\square$

We also need the following formulas, reported in [45], pg. 223, (2.0.1).

Proposition A7. Let ζ be the first hitting time of the level $a \in \mathbb{R}$ for a Brownian motion with drift μ . If $a > 0$, then

$$\mathbb{E}\left[e^{-y\zeta} I_{\zeta < \infty}\right] = e^{-\left(\sqrt{\mu^2+2y}-\mu\right)a}. \quad (\text{A8})$$

If $a < 0$, then

$$\mathbb{E}\left[e^{-y\zeta} I_{\zeta < \infty}\right] = e^{\left(\sqrt{\mu^2+2y}+\mu\right)a}. \quad (\text{A9})$$

Appendix C. Finite Difference Terms

$$\begin{aligned} F_t &= \frac{F(i-1, j) - F(i, j)}{\Delta t} \\ F &= \frac{F(i-1, j) + F(i, j)}{2} \\ F_x &= \frac{F(i-1, j) - F(i-1, j-1) + F(i, j) - F(i, j-1)}{2\Delta x} \\ F_{xx} &= \frac{F(i-1, j+1) - 2F(i-1, j) + F(i-1, j-1)}{2(\Delta x)^2} \\ &\quad + \frac{F(i, j+1) - 2F(i, j) + F(i, j-1)}{2(\Delta x)^2}. \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} 0 &= \frac{F(i-1, j) - F(i, j)}{\Delta t} + \\ &\quad + \frac{1}{2} r x_j \frac{F(i-1, j) - F(i-1, j-1) + F(i, j) - F(i, j-1)}{\Delta x} \\ &\quad + \frac{1}{4} \sigma^2 x_j^2 \left(\frac{F(i-1, j+1) - 2F(i-1, j) + F(i-1, j-1)}{(\Delta x)^2} + \frac{F(i, j+1) - 2F(i, j) + F(i, j-1)}{(\Delta x)^2} \right) \\ &\quad - \frac{1}{2} r (F(i-1, j) + F(i, j)). \end{aligned} \quad (\text{A11})$$

- If $j = l_i + 1$, then

$$\begin{aligned}
 & F(i, j) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_j}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_j^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
 & - F(i, j+1) \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \\
 & = F(i-1, j-1) \left(-\frac{1}{2} \frac{rx_j}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \right) \\
 & + F(i-1, j) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_j}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_j^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
 & + F(i-1, j+1) \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \\
 & - F(i, l_i) \left(\frac{1}{2} \frac{rx_j}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \right)
 \end{aligned} \tag{A12}$$

- If $l_i + 1 < j < k_i - 1$, then

$$\begin{aligned}
 & F(i, j-1) \left(\frac{1}{2} \frac{rx_j}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \right) \\
 & + F(i, j) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_j}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_j^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
 & - F(i, j+1) \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \\
 & = F(i-1, j-1) \left(-\frac{1}{2} \frac{rx_j}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \right) \\
 & + F(i-1, j) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_j}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_j^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
 & + F(i-1, j+1) \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2}.
 \end{aligned} \tag{A13}$$

- If $j = k_i - 1$, then

$$\begin{aligned}
 & F(i, j-1) \left(\frac{1}{2} \frac{rx_j}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \right) \\
 & + F(i, j) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_j}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_j^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
 & = F(i-1, j-1) \left(-\frac{1}{2} \frac{rx_j}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \right) \\
 & + F(i-1, j) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_j}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_j^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
 & + F(i-1, j+1) \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2} \\
 & + F(i, j+1) \frac{1}{4} \frac{\sigma^2 x_j^2}{(\Delta x)^2}.
 \end{aligned} \tag{A14}$$

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